Lecture 11: Area under a curve (without integration)

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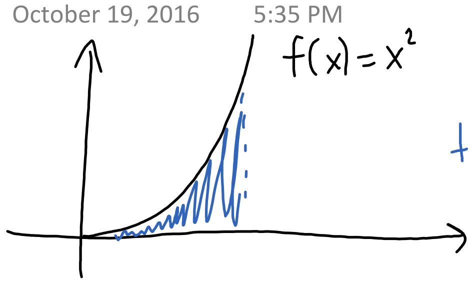
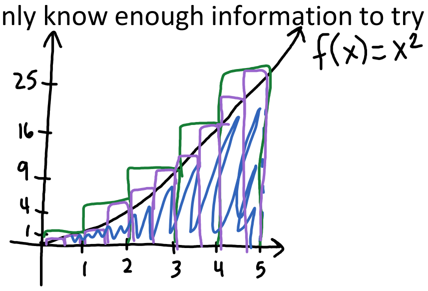


area under the bounds $x=0, x=5$

we only know enough information to try to approximate the area:



We want to compute the green area instead of the real area, even though this is clearly an over estimation.

Given area consists of rectangles of width 1, and height f(x).

$A_1 = 1(1) + 1(4) + 1(9) + 1(16) + 1(25)$

width height

$= 55$ ← first over-estimation of the area

<u>Better estimation</u>:

we need more rectangles for a better approximation:

- width of the new rectangles is now 1/2
- new points $x_1=1/2$, $x_2=1$, $x_3=3/2$,…, $x_9=9/2$, $x_{10}=5$
- height of rectangles is still the function value of x_i:

$$A_2 = \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}(1)^2 + \frac{1}{2}\left(\frac{3}{2}\right)^2 + \frac{1}{2}2^2 + \cdots + \frac{1}{2}\left(\frac{9}{2}\right)^2 + \frac{1}{2}(5)^2$$

$= 48.125$

<u>Next step</u>:

- intervals of 1/4, 1/8, …, infinitely we need limits!

In general, step size 1/n instead of 1/2…

- in general from 0 to 5 under curve $f(x)=x^2$
- width of rectangle: 1/n points $x_i = \frac{i}{n}(5-0) = \frac{5i}{n}$
- now area $= \frac{5}{n} * \left(\frac{5(1)}{n}\right)^2 + \frac{5}{n}\left(\frac{5(2)}{n}\right)^2 + \cdots \frac{5}{n}\left(\frac{5n}{n}\right)^2$
- Now let $n \to \infty$ for the most precise approximation:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5}{n}\left(\frac{5i}{n}\right)^2\right)$$

$$= \lim_{n\to\infty} \left(\frac{5}{n}\right) \sum_{i=1}^{n} \left(\frac{5i}{n}\right)^2$$

$$= \lim_{n\to\infty} \left(\frac{5}{n}\right) * \left(\frac{5}{n}\right)^2 \sum_{i=1}^{n} (i)^2$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n-1)}{6}$$

$$= \lim_{n\to\infty} \left(\left(\frac{5^3}{n^3}\right) * \frac{n(n+1)(2n-1)}{6}\right)$$

$$= \frac{5^3}{6} \lim_{n\to\infty} \left(\frac{2n^3 - n^2 + 2n^2 - n}{n^3} \right)$$

$$= \frac{5^3}{6} \lim_{n\to\infty} \left(\frac{\frac{1}{n^3}(2n^3 + n^2 - n)}{n^3\left(\frac{1}{n^3}\right)} \right)$$

$$= \frac{125}{6} \lim_{n\to\infty} \left(\frac{2 + \overset{\to 0}{\frac{1}{n}} - \overset{\to 0}{\frac{1}{n^2}}}{1} \right) = \frac{125}{6}(2) = \frac{125}{3} = 41.667$$

At n=2^{10}: we get 41.7277...
Here we have a formula for $\sum_{i=1}^{n}(i)^2$, but in general this is not possible (ie. next class we'll discover the proper way of computing the area under a curve).

In general:

Area under a curve of a function f(x) between x_0 and x_n is given as:

$$A = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

$$(x_1 = x_{i-1} + \Delta x)$$

The definite integral

$$\int_a^b f(x)dx = \lim_{n\to\infty} 1 \sum_{x=1}^{n} f(x_i \Delta x)$$

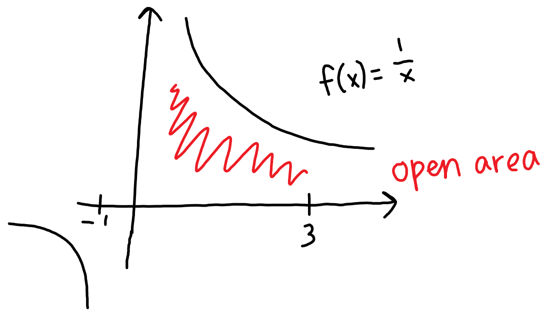
where x_0=a, x_n=b, $\Delta x = \frac{b-a}{n}$

If this limit exists, we say f is **integrable on [a,b]**.

Theorem:

If f is continuous on [a,b] (or has a finite number of jumps) then it is integrable on [a,b].

What is not integrable (finite area under curve)?



Properties:

$$\int_a^b f(x)\,\mathrm{dx} = -\int_a^b f(x)\,\mathrm{dx}$$

$$\int_a^a f(x)\,\mathrm{dx} = 0$$

$$\int_a^b \underset{\text{constant}}{c}\,\mathrm{dx} = c \cdot (b - a)$$

$$\int_a^b (f(x) \pm g(x))\,\mathrm{dx} = \int_a^b f(x)\,\mathrm{dx} \pm \int_a^b g(x)\,\mathrm{dx}$$

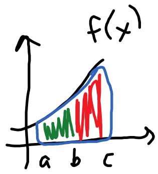
constant

b b

$$\int_a (f(x) \pm g(x))\, \mathrm{d}x = \int_a f(x)\, \mathrm{d}x \pm \int_a g(x)\, \mathrm{d}x$$

$$\int_a^b c \cdot f(x)\, \mathrm{d}x = \underset{\text{constant}}{c} \cdot \int_a^b f(x)\, \mathrm{d}x$$

$$\int_a^b f(x)\, \mathrm{d}x + \int_b^C f(x)\, \mathrm{d}x = \int_a^c f(\mathrm{x})\, \mathrm{d}x$$



Example: evaluate

$$\int_0^5 (4 + 3x^2)\, \mathrm{d}x$$

$$= \int_0^5 4\, \mathrm{d}x + \int_0^5 (3x^2)\, \mathrm{d}x$$

$$\int_0^5 4\, \mathrm{d}x = 4 \cdot \int_0^5 1 \cdot \mathrm{d}x = 4 \cdot 1(5-0)$$

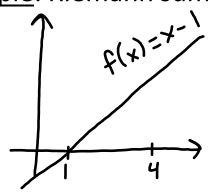
rule 1

$$= 4(5-0) + 3\int_0^5 x^2\, \mathrm{d}x$$

} first example of class value $\frac{125}{3}$

$$= 4(5) + 3\left(\frac{125}{3}\right) = 20 + 125 = \underline{\underline{145}}$$

Example: Riemann sum of f(x)=x-1 on [1,4]



$$R = \lim_{n\to\infty}\left(\sum_{i=0}^{n} f(x_i)\Delta x\right)$$

here $\Delta x = \frac{4-1}{n}$ ← width of interval, n ← # of steps

$\Rightarrow x_i = x_0 + i\Delta x$

$x_i = 1 + i\frac{3}{n} = 1 + \frac{3i}{n}$

$$\lim_{n\to\infty}\left(\sum_{i=0}^{n} f\left(1+\frac{3i}{n}\right)\cdot\frac{3}{n}\right) = \lim_{n\to\infty}\left(\sum_{i=0}^{n}\left(1+\frac{3i}{n}-1\right)\cdot\frac{3}{n}\right)$$

$$\lim_{n\to\infty}\left(\sum_{i=0}^{n} f\left(\frac{3i}{n}\right)\cdot\frac{3}{n}\right) = \lim_{n\to\infty}\left(\frac{9}{n^2}\right)\left(\sum_{i=0}^{n} i\right)$$

$$= \lim_{n\to\infty}\left(\frac{9}{n^2} * \frac{n(n+1)}{2}\right) = \frac{9}{2}\lim_{n\to\infty}\left(\frac{n(n+1)}{n^2}\right)$$ ← simplify

$$= \frac{9}{2}$$